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Discrete Applied Mathematics 143 (2004) 353–358
DISCRETE APPLIED MATHEMATICS
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# The cost of cutting out convex $n$-gons

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Received 29 April 2002; received in revised form 2 January 2003; accepted 24 September 2003

**Abstract**

Given a convex $n$-gon $P$ drawn on a piece of paper $Q$ of unit diameter we prove that it can be cut with a total cost of $O(\log n)$. This bound is shown to be asymptotically tight: a regular $n$-gon (whose circumscribed circle has radius, say, $1/3$) drawn on a square piece of paper of unit diameter requires a cut cost of $\Omega(\log n)$.


*Keywords:* Combinatorial geometry; Complexity of cutting; Convex polygons

## 1. Introduction

Overmars and Welzl [3] considered the following problem:

> Given a polygonal piece of paper $Q$ with a polygon $P$ drawn on it, cut $P$ out of the piece of paper in the cheapest possible way.

We are in the same framework as in [3]. A *cut* is a line that divides the piece of paper into a number of pieces, those that lie left of the line and those that lie right of the line (see Fig. 1). A cut is not allowed to intersect the interior of $P$. If the piece of paper is a concave polygon, one has to cut along all intervals where the cutting line intersects the paper. The pieces of paper resulting from a cut are the connected components of the paper minus the cutting line.

After a cut is made we continue with the piece of paper containing $P$. It is thus understood that if a cut generates multiple pieces, those but the one that contains $P$ are discarded. A *cutting sequence* is a sequence of cuts such that, after the last one, the piece of paper is the polygon $P$, as illustrated in Fig. 2. The cost of a cut is the length of the intersection of the cutting line with the (current) piece of paper containing $P$. The problem asks for a cutting sequence whose total cost is minimum. Such a sequence is called an *optimal cutting sequence*. The total cost of an optimal cutting sequence, denoted $h(P,Q)$, is called the *cut cost*. Note that $h(P,Q)$ not only depends on $P$ and $Q$, but also on the position of $P$ inside $Q$.

It is clear that the problem is solvable only if the polygon $P$ is convex, which we will assume. If the piece of paper $Q$ is convex, Overmars and Welzl [3] have shown that there exists an optimal cutting sequence for $P$ with $O(n)$ cuts, in which each cut touches polygon $P$, where $n$ is the number of edges of $P$ [3]. For the case of non-convex piece of paper $Q$, if $Q$ is considered a closed set (as opposed to open), there are examples in which no optimal cutting sequence exists. On the other hand, when $Q$ is non-convex and considered an open set, it is unknown whether an optimal cutting sequence exists. A more detailed discussion of these matters appears in [3].

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0166-218X/$ - see front matter © 2003 Published by Elsevier B.V.
doi:10.1016/j.dam.2003.09.001

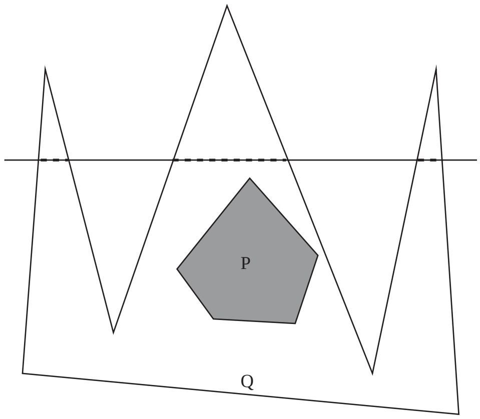


Fig. 1. A straight-line cut which generates four pieces of paper.

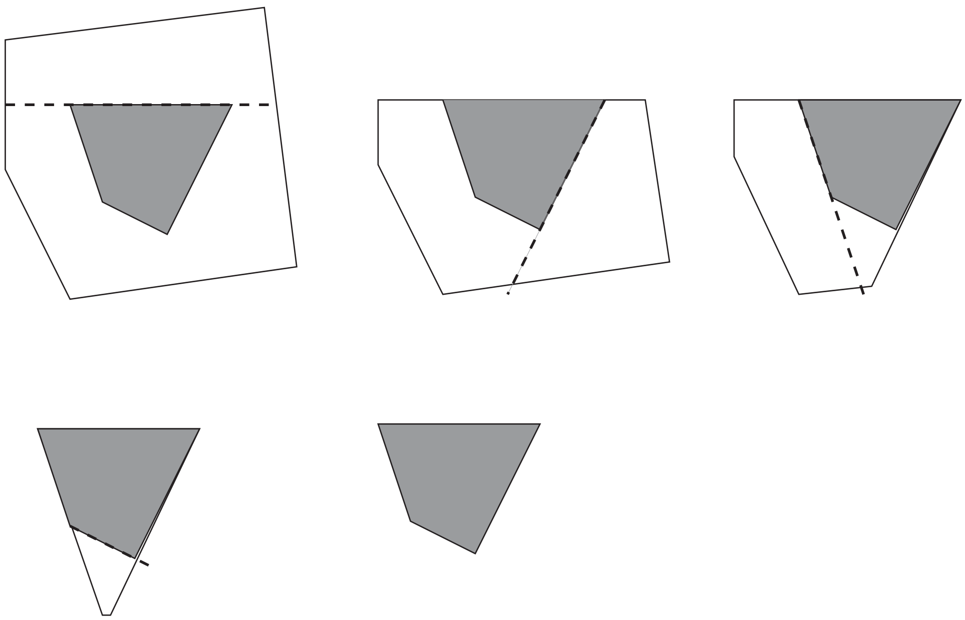

Fig. 2. A cutting sequence.

In this note we estimate the maximum cutting cost of a convex polygon with $n$ vertices drawn on a convex piece of paper of roughly the same size. Let

$$h(n) = \sup_{P \subset Q,\, diam(Q)=1} \{h(P,Q) \mid P \text{ is a convex } n\text{-gon, } Q \text{ is convex}\},$$

where $diam(Q)$ stands for the diameter of $Q$. (Clearly $h(n) \leqslant n$ exists, since cutting along each edge of $P$, in any order, gives a cutting sequence whose cost is not more than $n$.)

**Theorem 1.** $h(n) = \Theta(\log n)$.

The proof of the upper bound $h(n) = O(\log n)$ in the theorem also yields the following result. If $diam(Q)/diam(P) = O(1)$, there exists an $O(n \log n)$-time algorithm, which cuts out $P$ through a sequence of cuts of cost $O(|P| \log n)$, where $|P|$ denotes the perimeter of $P$. This fact has already been used to obtain an $O(\log n)$-approximation algorithm, which runs in $O(Nn + n \log n)$ time, for cutting out a convex $n$-gon $P$ out of a convex polygon $Q$ with $N$ sides [2].

We briefly mention two related problems. Pach and Tardos [4] have studied the problem of separating a large subfamily from a given family of pairwise disjoint compact convex sets on a sheet of glass, using the same type of line cuts. Recently, Demaine et al. [1] have given a characterization of the class of polygons that can be cut from a piece of material using a sufficiently small cutting segment (which models a circular saw).

## 2. Proof of Theorem 1

We first prove the upper bound using the following two-stage cutting algorithm.

*Stage* 1: Perform two parallel horizontal cuts touching $P$ from below and above. Then perform two parallel vertical cuts touching $P$ from the left and right. We can thus assume, without loss of generality, that $Q$ is the minimum axis-aligned rectangle containing $P$. Perform atmost eight additional edge cuts (along edges of $P$) at each vertex of $P$ contained in $\partial Q$ (the boundary of $Q$). The cost of the cuts done in Stage 1 is clearly O(1).

*Stage* 2: In this stage, each cut is along an edge of $P$ (i.e. we perform an edge cutting). Consider a set $\mathscr{T}$ of triangles, each corresponding to a convex arc of $P$ that remains to be cut. Initially $|\mathscr{T}| \leqslant 4$. For each triangle, two vertices are midpoints of edges of $P$ supporting consecutive edge cuts along $P$ and the third vertex is the intersection point of those cutting lines (see Fig. 3). We only include in $\mathscr{T}$ such triangles corresponding to pairs of edges that are themselves not adjacent in $P$. For each $\Delta \in \mathscr{T}$, we call the edge connecting the two midpoints of edges of $P$ the *base edge* of $\Delta$ (see Fig. 3). Each triangle in $\mathscr{T}$ is contained in the current piece of paper. Stage 2 proceeds in rounds, and the set $\mathscr{T}$ is updated during each round. At each round, the triangles in $\mathscr{T}$ satisfy the following properties:

- Each triangle corresponds to two consecutive cuts along $P$ (i.e. none of the edges of the convex arc between these edges have been cut).
- The interior angle of the triangle corresponding to the base edge is at least 90°.

Consider one such round. For each triangle $\Delta \in T$, perform a cut along the middle edge of the convex arc of $P$ contained in $\Delta$ (i.e. if the convex arc has $k$ edges numbered $1, 2, \ldots, k$, cut along edge $\lceil (k+1)/2 \rceil$ counting from one of its ends). Since the interior angle corresponding to the base edge of $\Delta$ is at least 90°, the length of the cut in $\Delta$ is bounded by the length of its base edge. We note that the endpoints of the base edges of the triangles in $\mathscr{T}$ form a convex polygon contained in $P$, so its perimeter is O(1). This implies that the total cost of the cuts in one round is also O(1). Each $\Delta \in T$ generates zero, one or two triangles to be included in $\mathscr{T}$ for the next round, while $\Delta$ is removed from $\mathscr{T}$. Clearly, the process terminates after atmost $\log n$ rounds (when cuts along all edges of $P$ have been done and $\mathscr{T} = \emptyset$), so the cost of the cuts done in Stage 2 is $O(\log n)$.

Since the cost of Stage 1 is O(1), the total cost of the cutting sequence produced by the algorithm is $O(\log n)$. As each round takes $O(n)$ time, the total time complexity is $O(n \log n)$.

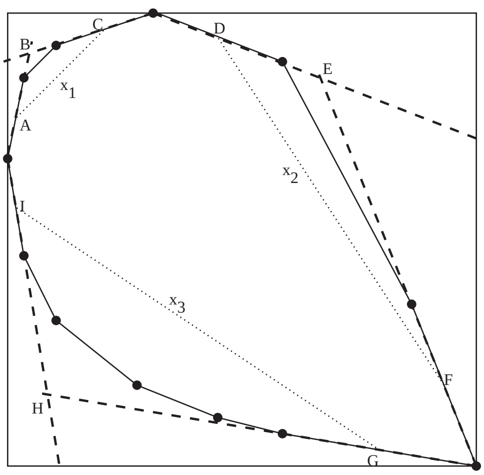


Fig. 3. Proof of the upper bound; triangles $ABC$, $DEF$ and $GHI$ are initially in $\mathscr{T}$; their base edges are $x_1, x_2, x_3$.

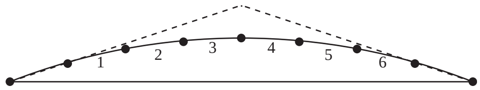


Fig. 4. A convex arc of $k = 6$ edges of $P$.

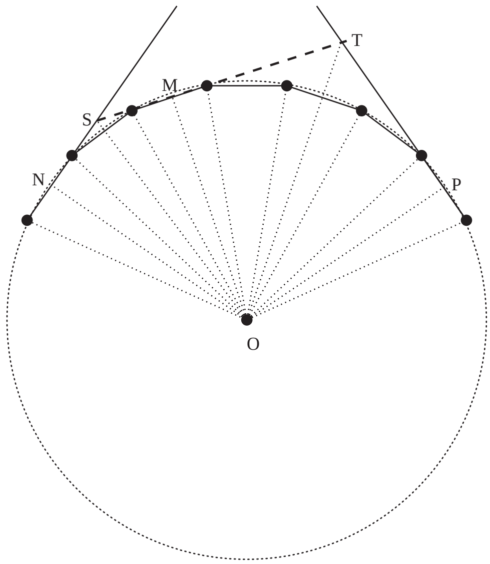


Fig. 5. Case 1 in the proof of the lower bound: an edge cut.

Next we prove the lower bound. Consider a regular $n$-gon $P$ inscribed in a circle of radius $r = 1/3$ drawn on a square piece of paper $Q$ of unit diameter (i.e. the diagonal of the square has unit length), so that $P$ and $Q$ have the same center $O$. We may assume that $n \geqslant 8$. First make two *free* cuts (whose cost is not counted) along two edges $e, f$ of $P$ so that the convex arc $C$ between $e$ and $f$ has $n/8$ uncut edges. Note that the angle between these cut-lines at their point of intersection and facing the polygon is larger than 90°. It is enough to prove a lower bound of $\Omega(\log n)$ on the cut cost of $C$.

Let $\alpha = 2\pi/n$ be the angle at $O$ between two adjacent points of $P$, and let $T(k)$ be the cut cost of a convex arc of $P$ consisting of $k$ (uncut) edges of $P$ (which we may assume is contained in a triangular piece of paper) (see Fig. 4). We will prove by induction on $k$ that $T(k) \geqslant c\alpha(k+1)\log(k+1)$ for $1 \leqslant k \leqslant n/8$. The lower bound then follows setting $k = n/8$:

$$T\left(\frac{n}{8}\right) \geqslant c\,\frac{2\pi}{n}\,\frac{n}{8}\log\frac{n}{8} = \Omega(\log n).$$

Denote by $l$ the side length of $P$: $l = 2r\sin(\alpha/2)$. The basis of the induction is satisfied by choosing $c$, as $T(i) \geqslant il$ for $i = 1, 2, 3, 4$, and $\sin(\alpha/2) > \alpha/4$ for $\alpha \leqslant \pi/4$. We use the following result on optimal cutting sequences. □

**Theorem 2** (Overmars and Welzl [3]). *If the piece of paper is convex, then there exists an optimal cutting sequence with* $O(n)$ *cuts in which each cut touches polygon* $P$, *where* $n$ *is the number of edges of* $P$.

There are two possible cases, as the first cut touching the arc may be an *edge cut* (i.e. along an edge of $P$), or a *vertex cut* (i.e. through a vertex of $P$).

*Case* 1: First cut touching the convex arc of $k$ edges is an edge cut. Refer to Fig. 5 for notation. Put $h = |ON| = |OM| = |OP| = r\cos(\alpha/2)$. The length of the cut is

$$z = |ST| = |SM| + |MT| = h\left(\tan\frac{i\alpha}{2} + \tan\frac{j\alpha}{2}\right)$$

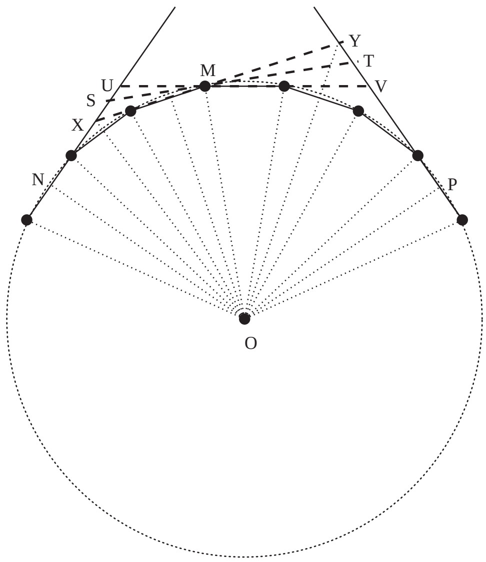


Fig. 6. Case 2 in the proof of the lower bound: a vertex cut.

for some $i, j \geqslant 0$ with $i+j=k+1$. We have $\cos(\alpha/2) > 1/2$ for $\alpha \leqslant \pi/4$. Using the inequality $\tan x > x$, for $x \in (0, \pi/2)$, we get that

$$z \geqslant \frac{r}{2} \min_{i+j=k+1} \left( \tan\frac{i\alpha}{2} + \tan\frac{j\alpha}{2} \right)$$

$$\geqslant \frac{r}{2} \min_{i+j=k+1} \left( \frac{i\alpha}{2} + \frac{j\alpha}{2} \right) = \frac{(k+1)\alpha}{12}.$$

After performing the first cut, we are left with two smaller convex arcs to be cut. This yields a cutting cost of at least

$$\frac{(k+1)\alpha}{12} + \min_{i+j=k-1} \{T(i) + T(j)\}.$$

*Case* 2: First cut touching the convex arc of $k$ edges is a vertex cut (through vertex $M$). Refer to Fig. 6 for notation. The length of the cut is

$$z = |ST| = |SM| + |MT| \geqslant |MU| + |MY|$$

$$= h\left(\tan\frac{i\alpha}{2} - \tan\frac{\alpha}{2}\right) + h\left(\tan\frac{j\alpha}{2} - \tan\frac{\alpha}{2}\right)$$

for some $i, j \geqslant 0$ with $i+j=k$. A similar calculation to the previous case (using the inequality $\tan x/2 < x$, for $x \in (0, \pi/2)$), gives

$$z \geqslant \frac{r}{2} \min_{i+j=k+2} \left( \frac{i\alpha}{2} - \alpha + \frac{j\alpha}{2} - \alpha \right) = \frac{(k-2)\alpha}{12}.$$

After performing the first cut, we do two more free cuts (whose cost is not counted) along the two edges of $P$ adjacent to $M$. We are left again with two smaller convex arcs to be cut. The corresponding cutting cost is at least

$$\frac{(k-2)\alpha}{12} + \min_{i+j=k-2} \{T(i) + T(j)\}.$$

Putting together Cases 1 and 2 we obtain the recurrence

$$T(k) \geqslant \frac{(k-2)\alpha}{12} + \min_{i+j=k-2} \{T(i) + T(j)\},$$

whose solution is

$$T(k) \geqslant c\alpha(k+1)\log(k+1), \quad \text{for some } c > 0.$$

This completes the proof of the theorem. □

## 3. Conclusion

It is natural to consider the following variant of the above problem in three dimensions.

Given a polyhedral piece of material $Q$, and a convex polyhedron $P$ contained in it, cut $P$ out of the piece of material in the cheapest possible way.

At this time, a cut is a plane that divides the piece of material into a number of pieces, and which is not allowed to intersect the interior of $P$. A cutting sequence is defined similarly to the planar case, so that after the last cut, the piece of material is the polyhedron $P$.

The cost of a cut is the *area* of the intersection of the cutting plane with the (current) piece of material containing $P$, and the problem asks for an optimal cutting sequence (i.e one whose total cost is minimum).

Similarly to the planar convex case, let

$$g(n) = \sup_{\substack{P \subset Q \\ d(Q)=1}} \{g(P,Q) \,|\, P \text{ is a convex polyhedron with } n \text{ vertices}, Q \text{ is convex}\},$$

where $d(Q)$ stands for the diameter of $Q$, and $g(P,Q)$ is the cost of cutting $P$ out of $Q$. What is the asymptotic growth rate of $g(n)$?

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